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A nonlinear theory for a flexible unsteady wing

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Abstract This paper extends the previous studies by Wu [Wu TY (2001) Adv Appl Mech 38:291–353; Wu TY (2005) Advances in engineering mechanics—reflections and outlooks. World Scientific; Wu TY (2006) Struct Control Health Monit 13:553–560] to present a fully nonlinear theory for the evaluation of the unsteady flow generated by a two-dimensional flexible lifting surface moving in an arbitrary manner through an incompressible and inviscid fluid for modeling bird/insect flight and fish swimming. The original physical concept founded by Theodore von Kármán and William R. Sears [von Kármán T, Sears WR (1938) J Aero Sci 5:379–390] in describing the complete vortex system of a wing and its wake in non-uniform motion for their linear theory is adapted and extended to a fully nonlinear consideration. The new theory employs a joint Eulerian and Lagrangian description of the wing motion to establish a fully nonlinear theory for a flexible wing moving with arbitrary variations in wing shape and trajectory, and obtain a fully nonlinear integral equation for the wake vorticity in generalizing Herbert Wagner's [Wagner H (1925) ZAMM 5:17–35] linear version for an efficient determination of exact solutions in general.

Keywords Arbitrary trajectory · Flexible wing · Nonlinear unsteady wing theory · Wake vorticity theorem

1 Introduction

In the world of self-locomotion of aquatic and aerial animals by using lifting surfaces such as wings and fins appended, there are several salient features of significance. First, the wings are in general large in aspect-ratio, a feature that would suit for an unsteady lifting-line approach. Secondly, the periodic flapping of the wing generally involves changes in surface-profile shape (or *shape function*), e.g. from a stretched–straight pronation in downward stroke to a form with an arched camber and spanwise bending in upward supination stroke. Further, in swift maneuvering, the wings may bend and twist asymmetrically to change and turn in orientation and trajectory, e.g. in the beautiful performance of a humming bird using a figure-eight wing

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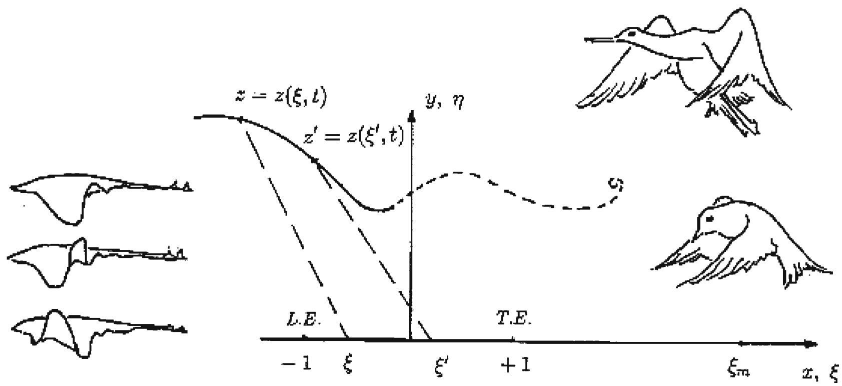

Fig. 1 The Lagrangian coordinates (ξ, η) adopted to describe arbitrary motion of a two-dimensional flexible lifting surface moving along arbitrary trajectory through fluid in an inertial frame fixed with the fluid at infinity

flapping in keeping its body fixed in front of a flower, and then suddenly fleeting off in a flash. All these features are so strongly nonlinear and time-varying that a comprehensively valid theory would have to take all these factors fully into account.

Recently, a nonlinear unsteady wing theory has been introduced by Wu [1–3] along this approach to provide an optimally unified analytical and numerical method for computation of solutions on specific premises. This nonlinear theory has been applied by Stredie [4] and Hou et al. [5,6] to perform computations of a number of unsteady motions of bodies shedding vortex sheet(s), attaining results of high accuracy (as measured versus relative errors and experiments available) in all the cases pursued. The present work is devoted to establishing a fully nonlinear theory for a two-dimensional flexible wing moving with arbitrary changes in wing shape and trajectory along the approach led by Wu [1–3] with intent to optimize the analytical and computational tasks required for attaining exact solutions efficiently.

2 Wing movement with arbitrary changes in shape and trajectory

We first recapitulate the nonlinear theory [1–3] of a two-dimensional arbitrary flexible lifting surface for modeling aquatic and aerial animal locomotion at high Reynolds number. We opt for two-dimensional theory for its simplicity to provide a foundation for further development of unsteady wing theory and for general applications.

In this respect, we find that of the existing linear theories, the simple and clear physical concept crystallized by von Kármán and Sears [7] in providing such a general view on an ingenious restructuring of the vorticity distribution over the wing and its trailing wake is readily found to afford powerful generalizations. So it has been extended by Wu [1, Sect 6] to account fully for all possible nonlinear effects in theory, and bring Herbert Wagner's pioneering work [8] to more general applications. The principal step is to employ a joint Eulerian and Lagrangian description of the lifting-surface movement for the formulation and analysis which we will delineate synoptically next. This useful description of unsteady bodily movement has also been applied by Lighthill [9] to develop a large-amplitude elongated-body theory.

Thus, we consider the irrotational flow of an incompressible and inviscid fluid generated by a two-dimensional flexible lifting surface $S_b(t)$ of negligible thickness, moving with time t through the fluid in arbitrary manner. Its motion can be described parametrically by using a Lagrangian coordinate system (ξ, η) to identify a point $X(\xi, t), Y(\xi, t)$ on the boundary surface $S(t) = S_b(t) + S_w(t)$ comprising the body surface S_b and a wake surface S_w, with $S(t)$ lying at time $t = 0$ over a stretch of the ξ-axis (at $\eta = 0$) and moving with time $t (\geq 0)$ as one that can be prescribed by complex variable $z = x + iy = Z(\xi, t)$ (see Fig. 1), parametrically in ξ as

$$Z(\xi,t) = X(\xi,t) + \mathrm{i}Y(\xi,t) \quad \text{on} \quad S_b(t)\colon (-1 < \xi < 1) + S_w(t)\colon (1 < \xi < \xi_m) \tag{1}$$

with the leading and trailing edges of the wing at $\xi = -1$ and $\xi = 1$, respectively. From the trailing edge a vortex sheet is assumed being shed smoothly (i.e., under the Kutta condition) to form a prolonging wake S_w, and ξ_m identifies the path $Z(\xi_m, t)$ of the starting vortex shed at $t = 0$ to reach $\xi_m = \xi_m(t)$ at time t. A simple choice for $(\xi + \mathrm{i}\eta)$ is the initial material position of $S_b(t = 0)$, taken to be in its stretched–straight shape such that $Z(\xi, 0) = \xi(-1 < \xi < 1, \eta = 0)$, lying in an unbounded fluid initially at rest in an *absolute* inertial frame of reference. The flexible $S_b(t)$ is assumed to be inextensible ($|Z_\xi| \equiv |\partial Z/\partial\xi| = 1$, or $X_\xi^2 + Y_\xi^2 = 1, |\xi| < 1$) and the point ξ on $S_b(t)$ moves with a *prescribed* (complex) velocity $W(\xi,t) = U - \mathrm{i}V$,

$$W(\xi,t) = U - \mathrm{i}V = \partial\overline{Z}/\partial t = X_t - \mathrm{i}Y_t \quad (|\xi| < 1, t \ge 0; \overline{Z} = X - \mathrm{i}Y), \tag{2}$$

which has a tangential component, $U_s(\xi,t)$, and a normal component, $U_n(\xi,t)$, given by

$$W\partial Z/\partial\xi = (X_\xi X_t + Y_\xi Y_t) - \mathrm{i}(X_\xi Y_t - Y_\xi X_t) = U_s - \mathrm{i}U_n, \tag{3}$$

and with the same expression for the wake surface $S_w(t)$ for $(1 < \xi < \xi_m)$.

In the spirit of von Kármán and Sears [7], we adopt for $t > 0$ the following vorticity distribution:

$$\begin{aligned} &\text{on } S_b(t)\colon && = \gamma(\xi,t) = \gamma_0(\xi,t) + \gamma_1(\xi,t) && (-1 < \xi < 1), \\ &\text{on } S_w(t)\colon && = \gamma(\xi,t) = \gamma_w(\xi,t) && (1 < \xi < \xi_m), \end{aligned}$$

where $\gamma_0(\xi,t)$ is the bound vortex distributed over S_b representing the "quasi-steady" flow past S_b such that the time t in the original prescribed $W(\xi,t)$ is frozen to serve merely as a parameter in evaluating the quasi-steady γ_0 (by steady airfoil theory), and $\gamma_1(\xi,t)$ is the additional bound vortex on S_b induced by the trailing wake vortex $\gamma_w(\xi,t)$ such that γ_1 and γ_w jointly bear no change to U_n (but not so to U_s) over S_b so as to reinstate the original time-varying normal velocity $U_n(\xi,t)$ prescribed on $S_b(t)$.

Thus, we represent the velocity field by a vorticity distribution, $\gamma(\xi,t)$, per unit length spanwise over the body and wake surfaces to give at time t the complex velocity $w(z,t) = u - \mathrm{i}v$ of the fluid at a field point z as

$$w(z,t) = \frac{1}{2\pi\mathrm{i}} \int_{-1}^{\xi_m} \frac{\gamma(\xi,t)}{Z(\xi,t) - z} \mathrm{d}\xi \quad (z = x + \mathrm{i}y \notin S, \ \ t \ge 0). \tag{4}$$

Applying Plemelj's formula to (4) yields for $w^\pm = \lim w(z(\xi + \mathrm{i}\eta), t)$ as $\eta \to \pm 0$ on the two sides of S as

$$u_s^\pm - \mathrm{i}u_n^\pm = w^\pm(\xi,t)\frac{\mathrm{d}Z}{\mathrm{d}\xi} = \pm\frac{1}{2}\gamma(\xi,t) + \frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}Z}{\mathrm{d}\xi}\int_S \frac{\gamma(\xi',t)}{Z' - Z}\mathrm{d}\xi' \quad (Z, Z' \in S, \ \ t \ge 0), \tag{5}$$

with $Z = Z(\xi,t), Z' = Z(\xi',t)$ both on S, and the integral assuming Cauchy's principal value. From (5),

$$\gamma(\xi,t) = u_s^+ - u_s^- \quad (-1 < \xi < 1 + \xi_m), \tag{6}$$

$$u_n^+(\xi,t) = u_n^-(\xi,t) = \mathfrak{Re}\left\{\frac{1}{2\pi}\frac{\mathrm{d}Z}{\mathrm{d}\xi}\int_S \frac{\gamma(\xi',t)}{Z' - Z}\mathrm{d}\xi'\right\}, \tag{7}$$

$$u_{sm} \equiv \frac{1}{2}(u_s^+ + u_s^-) = \mathfrak{Im}\left\{\frac{1}{2\pi}\frac{\mathrm{d}Z}{\mathrm{d}\xi}\int_S \frac{\gamma(\xi',t)}{Z' - Z}\mathrm{d}\xi'\right\}. \tag{8}$$

Here, (7) shows the continuity of normal velocity $u_n^+ = u_n^- = u_n$ across S and (8) gives the algebraic mean of the tangential velocity u_s on S. From (7)–(8) we deduce the contributions made separately by γ_0, γ_1, and

γ_w as:

$$U_n(\xi,t) = \Re\mathfrak{e}\left\{\frac{1}{2\pi}\frac{\mathrm{d}Z}{\mathrm{d}\xi}\int_{-1}^{1}\frac{\gamma_0(\xi',t)}{Z'-Z}\mathrm{d}\xi'\right\} \quad (Z = Z(\xi,t)\in S_b), \tag{9}$$

$$U_{1n}(\xi,t) = \Re\mathfrak{e}\left\{\frac{1}{2\pi}\frac{\mathrm{d}Z}{\mathrm{d}\xi}\int_{1}^{\xi_m}\frac{\gamma_w(\xi',t)}{Z'-Z}\mathrm{d}\xi'\right\} \quad (Z = Z(\xi,t)\in S_b), \tag{10}$$

$$-U_{1n}(\xi,t) = \Re\mathfrak{e}\left\{\frac{1}{2\pi}\frac{\mathrm{d}Z}{\mathrm{d}\xi}\int_{-1}^{1}\frac{\gamma_1(\xi',t)}{Z'-Z}\mathrm{d}\xi'\right\} \quad (Z = Z(\xi,t)\in S_b), \tag{11}$$

$$W_j(\xi,t) = U_j - \mathrm{i}V_j = \frac{1}{2\pi\mathrm{i}}\int_{S_b+S_w}\frac{\gamma(\xi',t)}{Z'-Z}\mathrm{d}\xi' \quad (Z = Z(\xi,t)\in S_w,\ \ \xi > 1), \tag{12}$$

where $W_j(\xi,t) = U_j - \mathrm{i}V_j$ is the jet-stream mean flow velocity averaged across the wake vortex sheet S_w which results from combining (7)–(8), giving, by (3), U_j and V_j as the *x*- and *y*-component of the wake fluid velocity on *Sw*.

The problem can now be recast to delineate the course for solution as follows. Equation 9 follows from invoking the condition that $u_n(\xi,t) = U_n(\xi,t)$, which is given at S_b, to give an integral equation for γ_0 which is to be solved, with time *t* frozen and without any unsteady wake, by applying steady airfoil theory. The velocity induced on S_b by the wake vorticity γ_w (while being transported with velocity W_j of the fluid particles on the wake) has the normal component U_{1n} given by (10), which is canceled out as is required of γ_1 on S_b according to (11), so that the sum (10)+(11) gives an integral equation for γ_1 in terms of γ_w. This solution for γ_1, which is to be determined under the Kutta condition (on the continuity of vorticity, and hence also the flow velocity bounded at the trailing edge) may be expressed, in principle, *symbolically* with a kernel $K(\xi';\xi,t)$ in the form

$$\gamma_1(\xi,t) = \int_1^{\xi_m} K(\xi';\xi,t)\gamma_w(\xi',t)\mathrm{d}\xi' \quad (|\xi|\le 1). \tag{13}$$

Finally, we apply Kelvin's theorem that the total circulation around $S_b + S_w$ must vanish $\forall t \ge 0$, i.e. $\Gamma_0 + \Gamma_1 + \Gamma_w = \int_{S_b}(\gamma_0+\gamma_1)\mathrm{d}\xi + \int_{S_w}\gamma_w\,\mathrm{d}\xi = 0$ (if it is zero initially), or, symbolically,

$$\Gamma_0 + \int_1^{\xi_m}\left\{1 + \int_{-1}^{1}K(\xi';\xi,t)\mathrm{d}\xi\right\}\gamma_w(\xi',t)\mathrm{d}\xi' = 0. \tag{14}$$

This is in essence the "generalized Wagner's integral equation" *formally* for wake vorticity γ_w. Its original linear version, attained by Wagner [8] and expounded by him and by von Kármán and Sears [7], has played a key role in finding the entire vorticity distributions and the final solutions of high accuracy to various linearized problems.

For the present nonlinear theory, it is of interest to derive the kernel $K(\xi,\xi',t)$ in closed form for efficient applications to wing movement in arbitrary manner. Such a desired integral equation has been first explicitly given by Wu [1], however in a rather lengthy series form by perturbation expansion. Another attempt has been made by Wu [3] to obtain an integral equation for γ_w by an integral iteration algorithm, with specific formulas for efficient computational application. The present work attempts to accomplish the theory fully generalized by including all possible nonlinear effects as fully as comprehended.

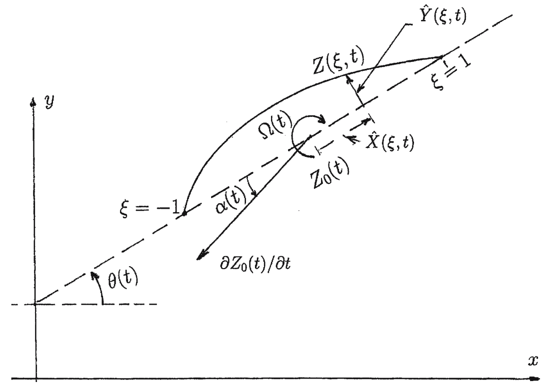

Fig. 2 The wing movement consists of (i) rectilinear translation with velocity $(\partial Z_0/\partial t)$ at incidence angle $\alpha(t)$, (ii) rotation with angular velocity $\Omega(t)$, and (iii) unsteady camber function $\hat{Z}(\xi,t)=\hat{X}+\mathrm{i}\hat{Y}$, $\hat{Y}=F(\hat{X},t)$

3 A unified method of solution

Here, the method proposed by Wu [1–3] based on a unified analytical-and-numerical scheme is further pursued to completion. Thus, following Wu [3], we first rewrite (9) as

$$U_n(\xi,t)=\frac{1}{2\pi}\int_{-1}^{1}\{1+g(\xi',\xi,t)\}\frac{\gamma_0(\xi',t)}{\xi'-\xi}\mathrm{d}\xi',$$
$$g(\xi',\xi,t)=\mathfrak{Re}\left\{\frac{\mathrm{d}Z}{\mathrm{d}\xi}\frac{\xi'-\xi}{Z'-Z}\right\}-1\quad(\forall(\xi,\xi')\in S_b).\tag{15}$$

As has been noted, if S_b is a flat wing, held at an arbitrary angle θ with the x-axis, we have, $\forall(\xi,\xi')\in S_b$,

$$Z(\xi)-Z(\xi')=\mathrm{e}^{\mathrm{i}\theta(t)}(\xi-\xi')\quad\longrightarrow\quad g(\xi',\xi,t)=0,\tag{16}$$

holding for arbitrary movement of the flat wing. For wings with arbitrary continuous camber, $g(\xi',\xi,t)$ is seen to be a regular function of $\xi,\xi'\forall t\geq 0$, especially in view of the characteristics that

$$g(\xi',\xi,t)=O(|\xi'-\xi|)\quad\text{as}\quad\xi'\to\xi,\tag{17}$$

and further is quadratic in the camber (see (22)). We can therefore call $g(\xi',\xi,t)$ the *residual kernel*, and its integral, the *residual integral*, which is of a form apt for iteration with rapid convergence.

For the body shape, S_b can always assume a *shape function* $Z(\xi,t)$ and a general *camber function* $\hat{Z}(\xi,t)\in C^1\forall\xi[-1,1]$ with respect to the 'absolute frame of reference' so that $Z(\xi,t)$ can be prescribed as

$$Z(\xi,t)=Z_0(t)+\mathrm{e}^{\mathrm{i}\theta}\hat{Z}(\xi,t)\quad(-1\leq\xi\leq 1),$$
$$\hat{Z}(\xi,t)=\hat{X}(\xi,t)+\mathrm{i}\hat{Y}(\xi,t)=\hat{X}(\xi,t)+\mathrm{i}F(\hat{X}(\xi,t),t),\tag{18}$$

(see Fig. 2). Here, $Z_0(t)$ is the origin of $\hat{Z}(\xi,t)$, $\theta(t)$ is the slope angle measured from the x-axis to the wing chord which passes through the leading edge at $Z(-1,t)$ and the trailing edge at $Z(+1,t)$ of the wing, and $\hat{Y}(\xi,t)=F(\hat{X}(\xi,t),t)$ stands for the real camber function, assumed regular. For convenience, we choose $Z_0(t)$ to be the projection of the wing central point $Z(0,t)$ onto the chord, i.e., by (18), with $\hat{X}(0,t)=0$,

$$Z_0(t)=Z(0,t)-\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\hat{Y}(0,t),\tag{19}$$

the leading and trailing edges at $\hat{Z}(-1,t)=-a(t)$, $\hat{Z}(1,t)=b(t)$, $c(t)=a(t)+b(t)$ being the chord length.

Next, for the inextensibility condition on the wing arc, we invoke $|\partial Z/\partial\xi|=|\partial\hat{Z}/\partial\xi|=1$, giving

$$\left\{1+\left(\frac{\partial F}{\partial\hat{X}}\right)^2\right\}^{1/2}\left|\frac{\partial\hat{X}}{\partial\xi}\right|=1.\tag{20}$$

For $\partial\hat{X}/\partial\xi > 0$, which is ordinarily the case, we have

$$\xi = \int_0^{\hat{X}} \left\{1 + \left(\frac{\partial F}{\partial \hat{X}}\right)^2\right\}^{1/2} \mathrm{d}\hat{X} \quad (-1 \le \xi \le 1), \tag{21}$$

from which follows $\hat{X}(\xi,t)$ by quadrature and inversion, and $\hat{Y}(\xi,t) = F(\hat{X}(\xi,t),t)$. In case of certain symmetry, e.g. a circular-arc wing, suitable parametric simplification may prevail for applications.

With $\hat{X}(\xi,t)$ and $\hat{Y}(\xi,t)$ so determined, the residual kernel becomes

$$g(\xi,\xi',t) = \frac{\hat{X}_\xi \Delta\hat{X} + \hat{Y}_\xi \Delta\hat{Y}}{(\Delta\hat{X})^2 + (\Delta\hat{Y})^2} - 1 \quad \left(\hat{X}_\xi \equiv \frac{\partial \hat{X}}{\partial \xi}, \Delta\hat{X} \equiv \frac{\hat{X}(\xi,t) - \hat{X}(\xi',t)}{\xi - \xi'}\right), \tag{22}$$

and similarly for $\hat{Y}_\xi$ and $\Delta\hat{Y}, (-1 \le \xi, \xi' \le 1)$. For a flat wing, $\hat{X}(\xi,t) = \xi, \hat{Y}(\xi,t) = 0$, hence $g(\xi,\xi',t) = 0$ as in (16). For wings of small camber, Eqs. (21)–(22) imply g being quadratic in the camber (maximum $|\hat{Y}|\forall t \ge 0$).

For given $Z(\xi,t)$ of $S_b(t)$, its surface (complex) velocity $W = U - \mathrm{i}V$ is, by (2) and (18),

$$W(\xi,t) = \frac{\partial}{\partial t}\overline{Z}(\xi,t) = \frac{\partial}{\partial t}(X - \mathrm{i}Y) = [(U_0 - \mathrm{i}V_0) + \mathrm{i}\Omega(\hat{X} - \mathrm{i}\hat{Y}) + (\hat{X}_t - i\hat{Y}_t)]\mathrm{e}^{-\mathrm{i}\theta}, \tag{23}$$

which prescribes the wing movement consisting in general of a translation with velocity $\mathrm{d}Z_0/\mathrm{d}t = (U_0 + \mathrm{i}V_0)\exp \mathrm{i}\theta$, which defines (U_0, V_0), a rotation of the wing chord about Z_0 with clockwise angular velocity $\Omega = -\mathrm{d}\theta/\mathrm{d}t$ (+ive for nose-up by convention), and a camber variation at the rate $(\hat{X}_t + \mathrm{i}\hat{Y}_t)\exp(i\theta)$. Hence by (3),

$$U_s(\xi,t) = (U_0 + \Omega\hat{Y} + \hat{X}_t)\hat{X}_\xi + (V_0 - \Omega\hat{X} + \hat{Y}_t)\hat{Y}_\xi, \tag{24}$$

$$U_n(\xi,t) = (V_0 - \Omega\hat{X} + \hat{Y}_t)\hat{X}_\xi - (U_0 + \Omega\hat{Y} + \hat{X}_t)\hat{Y}_\xi. \tag{25}$$

The normal velocity U_n of $S_b(t)$ will provide the kinematic flow condition (9) at S_b, and the tangential component U_s may serve to give the slip velocity between S_b and its adjacent fluid and to verify the wing being inextensible, if needed. So, with body motion (18) given, the surface velocity of $S_b(t)$ is completely determined.

After substituting the exact expression (25) for $U_n(\xi,t)$ in integral equation (15) for $\gamma_0(\xi,t)$, the leading term with the Cauchy kernel can be inverted by steady airfoil theory [1], i.e., letting $G_0\gamma_{00} = U_n(\xi,t)$, where

$$U_n(\xi,t) = \frac{1}{2\pi}\int_{-1}^{1} \frac{\gamma_{00}(\xi',t)}{\xi' - \xi}\mathrm{d}\xi' \equiv G_0\gamma_{00}, \longrightarrow \gamma_{00}(\xi,t) = G_0^{-1}U_n \quad (|\xi| < 1), \tag{26}$$

$$\gamma_{00}(\xi,t) = -\frac{2}{\pi}\sqrt{\frac{1-\xi}{1+\xi}}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\frac{U_n(\xi',t)}{\xi' - \xi}\mathrm{d}\xi' \equiv G_0^{-1}U_n, \tag{27}$$

where G_0 denotes the integral operator and G_0^{-1} its inverse (i.e., $G_0^{-1}G_0 = G_0G_0^{-1} = 1$) as designated. Applying this inversion to (15) in its entirety yields the following reduced integral equation for γ_0 as

$$\gamma_0(\xi,t) = \gamma_{00}(\xi,t) + H\gamma_0, \qquad \gamma_{00}(\xi,t) = G_0^{-1}U_n,$$

$$H\gamma_0 \equiv \int_{-1}^{1}\gamma_0(\xi',t)h(\xi',\xi,t)\mathrm{d}\xi', \qquad h(\xi',\xi,t) = \frac{1}{\pi^2}\sqrt{\frac{1-\xi}{1+\xi}}\int_{-1}^{1}\sqrt{\frac{1+\zeta}{1-\zeta}}\frac{g(\xi',\zeta,t)\mathrm{d}\zeta}{(\xi'-\zeta)(\zeta-\xi)}. \tag{28}$$

For a flat wing, the solution for γ_0 is $\gamma_0 = \gamma_{00} = G_0^{-1}U_n$ since $H\gamma_0 = 0$ due to $g = 0$ by (16). For cambered wings, γ_0 can be easily solved by iteration, using $\gamma_0^{(k)}(\xi,t) = \gamma_{00}(\xi,t) + H\gamma_0^{(k-1)}$, $(k = 1,2,\cdots)$ under the integral operator H, with $\gamma_0^{(0)} = 0$. In fact, the iteration by successive substitutions readily yields

$$\gamma_0(\xi,t) = (1 + H + H^2 + \cdots)\gamma_{00}(\xi,t) = (\textstyle\sum_{m=0}^{\infty} H^m)G_0^{-1}U_n, \tag{29}$$

provided that the process converges for prescribed time-varying camber distribution so that

$$\gamma_0(\xi,t) = G_0^{-1}(1+N_0)U_n(\xi,t) \quad (N_0 = G_0(\textstyle\sum_{m=1}^{\infty} H^m)G_0^{-1}) \tag{30}$$

in which N_0 is the nonlinear integral operator (just determined exactly by iteration) to represent the nonlinear effects due to the camber distribution which is instantaneously frozen at each time step. Being quadratic in the wing camber, N_0 vanishes for flat wing and its series expansion for cambered wings is expected to converge reasonably rapidly in general. This solution for γ_0 contributes a circulation $\Gamma_0(t)$ around the wing as

$$\Gamma_0(t) = \int_{-1}^{1} \gamma_0(\xi,t)\mathrm{d}\xi = -2\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}(1+N_0)U_n(\xi,t)\mathrm{d}\xi \tag{31}$$

in which the multi-integrals have all been reduced in number by one (with G_0^{-1} integrated out), leaving the term with N_0 to give the nonlinear non-stationary camber effects on $\Gamma_0(t)$.

For the wake-induced bound vortex γ_1, the complete analogy between (11) and (9) can be used to imply for γ_1 the solution which can first be written *formally* by analogy with (30) as

$$\gamma_1(\xi,t) = -G_0^{-1}(1+N_0)U_{1n}(\xi,t), \tag{32}$$

followed by having the unknown $U_{1n}(\xi,t)$ eliminated by applying (10) which we rewrite, like (15) for (9), as

$$U_{1n}(\xi,t) = \frac{1}{2\pi}\int_{1}^{\xi_m}\{1+g_1(\xi',\xi,t)\}\frac{\gamma_w(\xi',t)}{\xi'-\xi}\mathrm{d}\xi' \quad (|\xi|<1), \tag{33}$$

where $g_1(\xi',\xi,t)$ has the same expression as $g(\xi',\xi,t)$ of (15) but differs from it in range by having $Z(\xi,t) \in S_b$ but $Z' = Z(\xi',t) \in S_w$. As a result, unlike $g(\xi',\xi,t)$ being always small for S_b with a small camber, as shown by (22), $g_1(\xi',\xi,t)$ can become finite in magnitude, especially when S_b displaces itself by a finite amount at fast rate from a straight trajectory in the space. In such cases, the wake vortices can give rise to finite nonlinear effects on the flow field aside from the local nonlinear effects due to changes in body shape according to (15).

In general, substituting (33) for U_{1n} in (32), we can readily derive for the total circulation around the wing due to γ_1, $\Gamma_1 = \int_{S_b}\gamma_1(\xi,t)\mathrm{d}\xi$, to obtain the following result

$$\Gamma_1(t) = \int_{1}^{\xi_m}\left\{\sqrt{\frac{\xi+1}{\xi-1}} - 1 + N_w(\xi,t) + N_b(\xi,t)\right\}\gamma_w(\xi,t)\mathrm{d}\xi, \tag{34}$$

$$N_w(\xi,t) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}\frac{g_1(\xi,\xi',t)}{\xi-\xi'}\mathrm{d}\xi', \tag{35}$$

$$N_b(\xi,t) = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+\xi'}{1-\xi'}}N_0(\xi',t)\frac{1+g_1(\xi,\xi',t)}{\xi-\xi'}\mathrm{d}\xi'. \tag{36}$$

Finally, we apply Kelvin's theorem as we have expounded for (14) to obtain for γ_w the integral equation

$$\Gamma_0(t) + \int_{1}^{\xi_m}\left\{\sqrt{\frac{\xi+1}{\xi-1}} + N_w(\xi,t) + N_b(\xi,t)\right\}\gamma_w(\xi,t)\mathrm{d}\xi = 0. \tag{37}$$

This is the *general wake-vorticity theorem* expressed in terms of a *nonlinear wake-vorticity integral equation* for γ_w, which the wake vorticity γ_w must satisfy. It generalizes Wagner's integral equation for the linear case to account fully for a flexible wing in arbitrary movement. In this equation, $\Gamma_0(t)$ has a component in (31) with kernel $N_0(\xi,t)$ representing the local nonlinear effects on Γ_0 due to changes in body shape. In the wake integral, the term with $N_w(\xi,t)$ represents the nonlinear wake effects primarily due to finite changes in orientation and velocity of body movement. The other term with $N_b(\xi, t)$ represents the nonlinear effects due jointly to changes in body shape and their wake effects, since it vanishes, by (16), for a flat wing. In the linear limit, both N_w and N_b vanish, reducing (37) to Wagner's integral equation.

In computation, the motion of S_b is first prescribed for $t \geq 0$. In a small time interval δt_k at $t = t_k > 0$ $(k = 1, 2, \cdots)$, a new small segment $\delta\xi_k$ of S_w is created just downstream of the trailing edge (at $\xi = 1$). At the very first step, $k = 1$, invoking Kelvin's theorem on the variation $\delta\Gamma_0 = -\delta(\Gamma_1 + \Gamma_w)$ reduces (37) to yield

$$-\delta\Gamma_0(t) = \int_1^{1+\delta\xi_1} \left\{ \sqrt{\frac{\xi+1}{\xi-1}} + N_w(\xi,t) + N_b(\xi,t) \right\} \gamma_w(\xi,t)\mathrm{d}\xi \quad (\delta\xi_1 = |\overline{W}(1,t) + \overline{W}_j(1,t)|\delta t), \tag{38}$$

where $\overline{W}(1,t)\delta t$ is a segment of the prescribed trajectory traversed in δt by the trailing edge, and $\overline{W}_j(1,t)\delta t$ is the contribution by convection of (12) with the jet-stream leaving the trailing edge. For $0 < \delta\xi_1 \ll 1$, the wake vorticity $\gamma_w(\xi, t)$ shed continuously into $\delta\xi_1$ of S_w can be determined, by analysis and numerics, *accurately* from (38) by adapting method [7] of von Kármán–Sears to the present nonlinear case with iteration. Once this $\gamma_w(\xi, t_1)$ is determined, it will move on with the local fluid, keeping invariant in magnitude (by Helmholtz's theorem), with velocity $\overline{W}_j(\xi, t)$ for $\xi > 1, t > t_1$, starting from $\delta Z(\xi, t_1) = (\overline{W}(1,t) + \overline{W}_j(1,t))\delta t$ to its new position $\delta Z(\xi, t_k)$, while *en route* continuously inducing the new $\gamma_1(\xi, t_k)$ and hence the shedding of new wake vortex $\gamma_w(\xi, t_k)(k = 2, 3, \ldots)$. Because of the invariance in magnitude of γ_w after being shed, formula (38) will continue to hold $(\forall t = 2, 3, \ldots)$ with $\delta\xi_1$ in (38) replaced by $\delta\xi_2, \delta\xi_3, \ldots$, in successive time steps. Thus, the method for solving (37) consists of finding the analytical solution to (38) for the leading term, followed by numerical iterations for the higher-order terms. As a remark, if the shed $\gamma_w(\xi, t_k)$ is discretized, its reduced point vortex should be placed at its centroid in its birth grid for improved accuracy. The foregoing unified exposition of Wu [1–3] thus brings to completion the nonlinear theory for arbitrary wing movement.

4 Conclusions

In conclusion, we have addressed all the issues concerning the generation of entire vortex distribution over a flexible wing moving in arbitrary manner, with all the various nonlinear effects identified for general applications to self-propulsion and related studies. The final exact form (37) is based on series expansion of the residual integral to all orders in camber, its rapid convergence is expected (primarily due to the smallness of the kernel g as stressed by Wu [3]) and can be easily assessed in practice for the accuracy of the contributions from the nonlinear effects in increasing orders, as should be straightforward by computation. These nonlinear effects are expected to play an active and important role in aerial and aquatic animal locomotion.

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